Mathematical Analysis of an SIR Disease Model With Non-Constant Transmission Rate

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Introduction

- Epidemiology: A branch of medicine that studies causes, transmission, and control methods of diseases at the population level.
- Mathematical epidemiology deals with creating a model for a disease through the study of incidence and distribution of the disease throughout a population.
- Here, we have examined the behavior of a measles-like disease[2] that is characterized by a non-constant transmission rate.

Definitions

State Variables

- S(t): Number of susceptible individuals at time t
- I(t): Number of infectious individuals at time t
- R(t): Number of recovered individuals at time t

Parameters:

- β: Transmission Rate constant
- α: Recovery Rate
- Λ: Birth Rate
- μ: Natural Death Rate
- ν: Proportionality constant

Other Useful Definitions

- Incidence: The number of individuals who become infected per unit time.
- Reproduction Number ($\mathcal{R}_0$): A useful threshold quantity that can be interpreted as the number of secondary infections generated by a single infectious individual in an entirely susceptible population during their infectious period. [3]

Model

The model we worked with comes from the Chapter 3 Problems found in "An Introduction to Mathematical Epidemiology" by Maia Martcheva [3]. This model is similar to the basic SIR Model proposed by Kermack and McKendrick [1], however this model features a non-constant transmission rate, $\beta(1+\nu I)$, which varies linearly with the size of the Infected class, I.

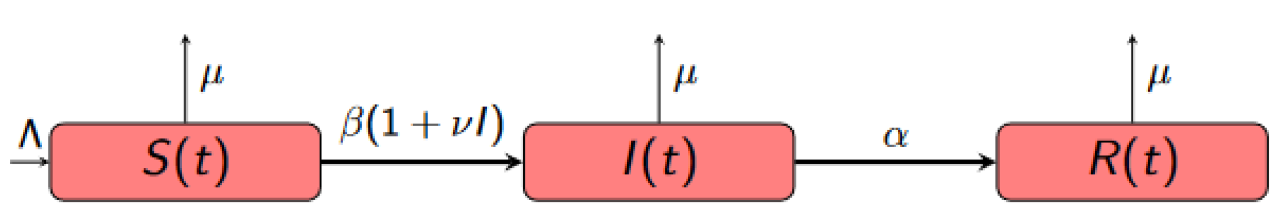

Fig. 1: Flowchart depicting movement between classes

The flowchart above can be represented mathematically by a system of nonlinear ordinary differential equations.

$$\begin{cases} S'(t) &= \Lambda - \beta(1+\nu I)IS - \mu S \\ I'(t) &= \beta(1+\nu I)IS - (\alpha+\mu)I \\ R'(t) &= \alpha I - \mu R \end{cases} \quad (1)$$

Because S' and I' have no dependence on R we can simplify the above model:

$$S'(t) = \Lambda - \beta(1+\nu I)IS - \mu S \quad (2)$$
$$I'(t) = \beta(1+\nu I)IS - (\alpha+\mu)I \quad (3)$$

Reproduction Number and Equilibria

The Reproduction number can be found in multiple ways, but the simplest way to show its derivation is by using the physical interpretation of $\mathcal{R}_0$

For our model, $\mathcal{R}_0 = \dfrac{\beta\Lambda}{\mu(\alpha+\mu)}$

We then find our disease-free and endemic equilibrium points by setting $S' = 0, I' = 0$, then solving our equilibrium equations for I^*:

$$I^*\Big[\beta(1+\nu I^*)S^* - (\alpha+\mu)\Big] = 0$$

- Disease-Free Equilibrium (DFE): $I^* = 0 \implies S^0 = \frac{\Lambda}{\mu}, I^0 = 0.$
- Endemic Equilibrium (EE) *see next section*: $I^* \neq 0 \implies \big[\beta(\alpha+\nu I^*)S^* - (\alpha+\mu)\big] = 0.$

Stability Analysis

We examined the stability of the DFE using a process called linearization. Linearization allows us to analyze behavior near the equilibrium by using perturbations, or small disturbances from the equilibrium. The linearized system is shown here in terms of the perturbations $u(t) = S(t) - S^*$ and $v(t) = I(t) - I^*$.

$$u'(t) = \frac{\partial S'}{\partial S}u(t) + \frac{\partial S'}{\partial I}v(t)$$
$$v'(t) = \frac{\partial I'}{\partial S}u(t) + \frac{\partial I'}{\partial I}v(t)$$

which can be solved for

$$u(t) = \bar{u}e^{\lambda t}$$
$$v(t) = \bar{v}e^{\lambda t}$$

For $\lambda < 0$, any disturbance in the system results in the system returning to equilibrium, and the system is said to be stable.

$$J = \begin{pmatrix} \frac{\partial S'}{\partial S} & \frac{\partial S'}{\partial I} \\ \frac{\partial I'}{\partial S} & \frac{\partial I'}{\partial I} \end{pmatrix} = \begin{pmatrix} -\nu\beta I^2 - \beta I - \mu & -2\nu\beta SI - \beta S \\ \nu\beta I^2 + \beta I & 2\nu\beta IS + \beta S - \alpha - \mu \end{pmatrix}$$

When we examine the Jacobian at the DFE we get:

$$J^0 = \begin{pmatrix} -\mu & -\frac{\beta\Lambda}{\mu} \\ 0 & \frac{\beta\Lambda}{\mu} - \alpha - \mu \end{pmatrix}$$

We now look at the instability of the DFE when $\mathcal{R}_0 > 1$. Using the Characteristic Equation Approach, we set the determinant of $J^0 - \lambda I$ equal to 0:

$$|J^0 - \lambda I| = 0 \implies \lambda^2 + \Big(2\mu + \alpha - \frac{\beta\Lambda}{\mu}\Big)\lambda + \mu(\mu+\alpha) - \beta\Lambda = 0$$

...applying the quadratic formula to solve for λ...

$$\lambda = \frac{-(2\mu + \alpha - \frac{\beta\Lambda}{\mu}) \pm \sqrt{(2\mu+\alpha-\frac{\beta\Lambda}{\mu})^2 - 4(\mu(\mu+\alpha) - \beta\Lambda)}}{2}$$

Since $\mathcal{R}_0 = \frac{\beta\Lambda}{\mu(\mu+\alpha)} > 1$,

$$\beta\Lambda > \mu(\mu+\alpha) \implies \mu(\mu+\alpha) - \beta\Lambda < 0 \implies -4(\mu(\mu+\alpha) - \beta\Lambda) > 0$$

Now we can see clearly that there exists at least one positive real eigenvalue λ. This shows that the DFE is unstable when $\mathcal{R}_0 > 1$.

Theorem 1. *For this model, DFE is stable whenever $\mathcal{R}_0 < 1$ and unstable when $\mathcal{R}_0 > 1$.*

Uniqueness of Endemic Equilibrium

Now we shall examine the uniqueness of the Endemic Equilibrium.
Starting with our equilibrium equations:

$$0 = \Lambda - \beta(1+\nu I^*)I^*S^* - \mu S^* \quad (4)$$
$$0 = \beta(1+\nu I^*)I^*S^* - (\alpha+\mu)I^* \quad (5)$$

We can solve (5) for S^* to arrive at

$$S^* = \frac{\alpha+\mu}{\beta(1+\nu I^*)}$$

Plug S^* into (4) and rearrange to arrive at a equation that is quadratic in I^*.

$$\beta\nu(\alpha+\mu){I^*}^2 + \beta((\alpha+\mu) - \Lambda\nu)I^* + (\mu(\mu+\alpha) - \Lambda\beta) = 0$$

Then, using the Quadratic Formula to solve for I^*, and, I^* being a positive real number, only considering the positive sign

$$I^* = \frac{-\beta(\alpha+\mu-\Lambda\nu) + \sqrt{\beta^2(\alpha+\mu-\Lambda\nu)^2 - 4(\beta\nu(\mu+\alpha))(\mu(\mu+\alpha) - \Lambda\beta)}}{2\beta\nu(\mu+\alpha)}$$

Because those in blue are all positive constants, we can see that when $\mathcal{R}_0 > 1$

$$\mathcal{R}_0 = \frac{\beta\Lambda}{\mu(\mu+\alpha)} > 1 \implies (\mu(\mu+\alpha) - \Lambda\beta) < 0 \implies 4ac < 0$$

and

$$\sqrt{\beta^2(\alpha+\mu-\Lambda\nu)^2 - 4(\beta\nu(\mu+\alpha))(\mu(\mu+\alpha) - \Lambda\beta)} > \beta(\alpha+\mu-\Lambda\nu)$$

Thus, I^* is both positive and unique.

Theorem 2. *When $\mathcal{R}_0 > 1$, we have a unique endemic equilibrium*

Future Work

- Continuing stability analysis on EE
- Forming our own model for COVID-19 dynamics in Oklahoma
- Finding numerical values for given parameters by fitting to data

References

[1] W.O. Kermack and A.G. McKendrick. "A contribution to the mathematical theory of epidemics". In: *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character* 115.772 (Aug. 1927), pp. 700–721. DOI: 10.1098/rspa.1927.0118.

[2] Xinzhi Liu and Peter Stechlinski. "Infectious disease models with time-varying parameters and general nonlinear incidence rate". In: *Applied Mathematical Modelling* 36.5 (2012), pp. 1974–1994. DOI: 10.1016/j.apm.2011.08.019.

[3] Maia Martcheva. *An Introduction to Mathematical Epidemiology*. 1st ed. Springer, 2015.