Isolated point theorems for uniform algebras on smooth manifolds

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by

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Abstract. In 1957, Andrew Gleason conjectured that if A is a uniform algebra on its maximal ideal space X and every point of X is a one-point Gleason part for A, then A must contain all continuous functions on X. Gleason's conjecture was disproved by Brian Cole in 1968. In this paper, we establish a strengthened form of Gleason's conjecture for uniform algebras generated by real-analytic functions on compact subsets of real-analytic three-dimensional manifolds-with-boundary.

1. Introduction. Let X be a compact Hausdorff space and $C(X)$ be the algebra of all complex-valued continuous functions on X with the supremum norm $\|f\| = \sup\{|f(x)| : x \in X\}$. A *uniform algebra* A on X is a uniformly closed subalgebra of $C(X)$ that separates the points of X and contains the constant functions on X. In 1957, Andrew Gleason [11] conjectured that if A is a uniform algebra on its maximal ideal space X and every point of X is a one-point Gleason part for A, then $A = C(X)$. Gleason's conjecture was disproved by Brian Cole in 1968 [6] (or see [5, Appendix], [17, Section 19]). Nevertheless, in this paper, we establish a strengthened form of Gleason's conjecture for uniform algebras generated by real-analytic functions on compact subsets of real-analytic three-dimensional manifolds. We assume that every point of X is isolated in the dual space norm on A^*. This condition is weaker than the original hypothesis "every point of X is a one-point Gleason part for A" in Gleason's conjecture. More precisely, the statement of the main result is as follows.

Theorem 1.1. *Let M be a real-analytic three-dimensional manifold-with-boundary. Assume that X is a compact subset of M such that the boundary ∂X of X relative to M is a two-dimensional submanifold of class C^1. Let A be a uniform algebra on X generated by a collection $\mathcal{F}$ of functions that are*

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real-analytic on a neighborhood of X. If

(i) *the maximal ideal space of A is X, and*
(ii) *every point of X is isolated in the dual space norm on A^*,*

then $A = C(X)$.

In the above theorem, by the *boundary ∂X of X relative to M* we mean the union of the topological boundary of X relative to M and the set $X \cap \partial M$.

The isolated point theorem stated above strengthens the peak point theorem [2, Theorem 1.1] proved by John Anderson and Alexander Izzo. In the same setting, they showed that if the maximal ideal space of A is X and every point of X is a peak point for A, then $A = C(X)$.

We also establish another isolated point theorem different from the preceding one in nature.

THEOREM 1.2. *Let M be a compact real manifold-with-boundary of class C^2. Let A be a uniform algebra on M generated by functions of class C^2. If*

(i) *the maximal ideal space of A is M, and*
(ii) *every point of M is isolated in the dual space norm on A^*,*

then the essential set for A has empty interior in M. Hence, in particular, the uniform algebra A is not essential.

Note that Theorem 1.2 requires C^2 smoothness only and applies to real manifolds-with-boundary of all dimensions. In addition, the conclusion implies that the functions in the uniform algebra A are arbitrary except on a small subset of M.

In the next section, we discuss the necessary background. We then prove preliminary results and important lemmas in Section 3. Finally, we prove both the isolated point theorems in Section 4.

2. Background. Let A be a uniform algebra on X. A well-known necessary (but not sufficient) condition for $A = C(X)$ is that the maximal ideal space $\mathfrak{M}_A$ of A is X (equivalently, the only nonzero multiplicative linear functionals on A are the point evaluations at points of X). There are other necessary conditions for $A = C(X)$ involving peak points, point derivations, Gleason parts, and isolated points (in the dual space norm on A^*). A point x in X is called a *peak point* for A if there exists f in A such that $f(x) = 1$ and $|f(y)| < 1$ for all y in $X \setminus \{x\}$. A (*bounded*) *point derivation* at a point ϕ in $\mathfrak{M}_A$ is a (bounded) linear functional $d\colon A \to \mathbb{C}$ that satisfies the Leibniz rule: $d(fg) = d(f)\phi(g) + \phi(f)d(g)$ for f, g in A. The *Gleason parts* for A are the equivalence classes under the equivalence relation $\sim$ on $\mathfrak{M}_A$ defined by $\phi \sim \psi$ if and only if $\|\phi - \psi\|_A < 2$ (see [5, Theorem 2.6.3]), where $\|\cdot\|_A$

is the dual space norm on A^*. By an *isolated point* for A, we will mean an isolated point in the dual space norm on A^*.

Now, for an arbitrary point p in X, consider the following four statements:

(a) p is a peak point for A,
(b) there is no nonzero point derivation on A at p,
(c) p is a one-point Gleason part for A,
(d) p is an isolated point (in the dual space norm on A^*) for A.

We can show that (a)⇒(b)⇒(d) (using [5, Corollary 1.6.7] and [5, Theorem 1.6.2] respectively) and (a)⇒(c)⇒(d) (the first implication follows easily, and the second one is obvious). The reverse implications in both the cases are false in general (for counterexamples see [6], [15, Example 5.13], [9], and [17, §18]). It easily follows that if $A = C(X)$, then each of the statements (b), (c), and (d) holds for all points p in X. In addition, if X is metrizable, then $A = C(X)$ implies that the statement (a) holds for all points p in X. In 1957, Gleason [11] conjectured that if the maximal ideal space of A is X and the statement (c) holds for all points p in X, then $A = C(X)$. More explicitly, the following conjecture was made.

Conjecture 2.1 (Gleason's conjecture). *If the maximal ideal space of A is X and every point of X is a one-point Gleason part for A, then $A = C(X)$.*

Subsequently, the following two related conjectures were considered.

Conjecture 2.2 (Peak point conjecture). *If the maximal ideal space of A is X and every point of X is a peak point for A, then $A = C(X)$.*

Conjecture 2.3 (Point derivation conjecture). *If the maximal ideal space of A is X and there is no nonzero point derivation for A, then $A = C(X)$.*

In this paper, we also consider the following stronger conjecture.

Conjecture 2.4 (Isolated point conjecture). *If the maximal ideal space of A is X and every point of X is isolated for A, then $A = C(X)$.*

A counterexample produced by Cole disproved all four conjectures [6] (or see [5, Appendix], [17, Section 19]). A few years later, a simpler counterexample was given by Richard Basener [3] (or see [17, Example 19.8]). Nevertheless, in 2016, the author [10] showed that the isolated point conjecture, the strongest of all four conjectures, holds for two important classes of uniform algebras, namely, uniform algebras generated by smooth functions on compact smooth two-dimensional manifolds and uniform algebras generated by polynomials on compact subsets of real-analytic three-dimensional submanifolds of complex Euclidean spaces.

Extending their previous work on the peak point conjecture, in 2009, Anderson and Izzo established a peak point theorem for uniform algebras generated by real-analytic functions on compact subsets of real-analytic three-dimensional manifolds-with-boundary [2, Theorem 1.1]. In this paper, we will establish the isolated point conjecture for the same class of uniform algebras. We note that the main result (Theorem 1.1) contains the peak point theorem [2, Theorem 1.1] as well as the isolated point theorem [10, Theorem 3.1].

In a recent paper [13], Izzo proved the following result.

THEOREM 2.5 ([13, Theorem 1.1]). *Let V be a real-analytic subvariety of an open set $\Omega \subset \mathbb{R}^n$, and let X be a compact subset of V such that ∂X is a real-analytic subvariety of V. Let A be a uniform algebra on X generated by a collection $\mathcal{F}$ of functions real-analytic on X. If*

(i) *the maximal ideal space of A is X, and*
(ii) *the set X does not contain any analytic disc,*

then $A = C(X)$.

Note that in our main result the boundary ∂X is a manifold of class C^1, but it is not necessarily a real-analytic subvariety. Therefore, our main result is of a different nature and cannot be obtained from the above-mentioned theorem.

3. Preliminaries and lemmas. The proofs of our results will use ideas from the work by Anderson and Izzo [2]. We will also use the following lemma proved in [10].

LEMMA 3.1 ([10, Lemma 2.4]). *Let A be a uniform algebra on X and Y be a closed subset of X.*

(i) *If a point in Y is isolated for A, then it is also isolated for $\overline{A|Y}$, the uniform closure of the algebra $A|Y = \{f|Y \in C(Y) : f \in A\}$.*
(ii) *Let $\tilde{B}$ be a uniform algebra on Y containing $A|Y$ and with maximal ideal space Y. If a point in Y is isolated for A, then it is also isolated for $\tilde{B}$.*

Let A be a uniform algebra on a compact space X. The *essential set* for A, a notion introduced by Herbert Bear, is the unique minimal closed subset $\mathcal{E}$ of X with the property that A contains every continuous function on X that vanishes on $\mathcal{E}$ [4, §2] (or see [5, Theorem 2.8.1]). In other words, $\mathcal{E}$ is the unique minimal closed subset of X such that A contains every continuous function whose restriction to $\mathcal{E}$ lies in $A|\mathcal{E}$. Bear proved that $A|\mathcal{E}$ is uniformly closed (and hence forms a uniform algebra) [4, Theorem 2]. More importantly, he showed that the maximal ideal space of $A|\mathcal{E}$ is $\mathcal{E}$ if and only if the maximal ideal space of A is X [4, Theorem 4]. The author strengthened these results in the form of the following theorem.

THEOREM 3.2 ([10, Theorem 2.5]). *Suppose A is a uniform algebra on X and L is a closed subset of X containing the essential set for A. Then $A|L$ is* *uniformly closed in $C(L)$. Moreover, the maximal ideal space of $A|L$ is L if and only if the maximal ideal space of A is X.*

We will use the following result by Izzo [14] in order to reduce approximation on a compact set to approximation on a smaller subset.

THEOREM 3.3 ([14, Theorem 1.3]). *Let A be a uniform algebra on X. Suppose that the maximal ideal space of A is X. Suppose also that E is a closed subset of X such that $X \setminus E$ is an m-dimensional manifold and such that*

(i) *for each point p in $X \setminus E$, there are functions f_j in A ($j = 1, \ldots, m$) that are C^1 on $X \setminus E$ and satisfy $df_1 \wedge \cdots \wedge df_m(p) \neq 0$, and*
(ii) *the functions in A that are C^1 on $X \setminus E$ separate the points of X.*

Then $A = \{g \in C(X) : g|E \in A|E\}$, i.e., the essential set for A is contained in E.

For a uniform algebra A on a certain compact set K, the following result of Anderson and Izzo provides a sufficient condition for $A = C(K)$. The n-dimensional Hausdorff measure is denoted by $\mathcal{H}^n$. For the notion of Hausdorff measure in a metric space and related results, see [7].

LEMMA 3.4 ([2, Lemma 2.1]). *Let K be a compact metric space and A be a uniform algebra on K with maximal ideal space K. Suppose $K = X \cup Y$ where X is a compact set such that $\overline{A|X} = C(X)$ and Y is a set with $\mathcal{H}^2(Y) = 0$. Suppose also that A is generated by Lipschitz functions. Then $A = C(K)$.*

Let A be a uniform algebra on X with maximal ideal space $\mathfrak{M}_A$ and f be a function in A. The *Gelfand transform* of f is the complex-valued function $\hat{f}$ on $\mathfrak{M}_A$ defined by $\hat{f}(\phi) = \phi(f)$ for $\phi \in \mathfrak{M}_A$. The collection $\hat{A} = \{\hat{f} : f \in A\}$ is a uniform algebra on $\mathfrak{M}_A$ (see [17, p. 37]). We let $\mathbb{B}(x, r)$ be the open ball in $\mathfrak{M}_A$ with center x and radius r, and let $\mathbb{S}(x, r)$ be the boundary of $\mathbb{B}(x, r)$ in $\mathfrak{M}_A$.

The following lemma is a generalization of [1, Corollary 21.10].

LEMMA 3.5. *Let A be a uniform algebra on K such that the maximal ideal space $\mathfrak{M}_A$ of A is metrizable with metric d. Assume that the uniform algebra $\hat{A}$ is generated by a collection of Lipschitz functions on $\mathfrak{M}_A$. Then for $x \in L = \mathfrak{M}_A \setminus K$ and $r > 0$, the two-dimensional Hausdorff measure satisfies $\mathcal{H}^2(L \cap \mathbb{B}(x, r)) > 0$.*

For a proof of the above lemma, we first define the following notion. Let A be a uniform algebra on X and K be a closed subset of X. The *A-convex hull* $\hat{K}$ of K is defined as the set

$$\hat{K} = \{\phi \in \mathfrak{M}_A : |\phi(f)| \leq \|f\| \text{ for all } f \text{ in } A\}.$$

Moreover, K is *A-convex* if $\hat{K} = K$. The A-convex hull $\hat{K}$ of K can be viewed as the maximal ideal space of the uniform algebra $\overline{A|K}$ [8, Ch. II, Theorem 6.1].

We also need the following generalization of [16, Theorem 1.6.2].

Lemma 3.6. *Let X be a compact metric space and A be a uniform algebra on X generated by a collection of Lipschitz functions. If S is a closed subset of X with $\mathcal{H}^1(S) = 0$, then $\overline{A|S} = C(S)$ and S is A-convex.*

Proof. Note that we only need to show $\overline{A|S} = C(S)$. Using the Stone–Weierstrass theorem, it is then sufficient to show that the real-valued functions in $\overline{A|S}$ separate the points of S. Fix two distinct points x and y in S. Since A is generated by a collection of Lipschitz functions, there is a Lipschitz function $f \in A$ such that $f(x) \neq f(y)$. Since $\mathcal{H}^1(S) = 0$ and f is a Lipschitz function, the one-dimensional Hausdorff measure $\mathcal{H}^1(f(S))$ equals 0. Then by [7, Corollary 2.10.12], $f(S)$ is totally disconnected. Therefore, there exist disjoint open subsets U and V of $\mathbb{C}$ such that $f(x) \in U$, $f(y) \in V$, and $f(S) \subseteq U \cup V$. Define $\phi\colon U \cup V \to \mathbb{R}$ by $\phi(u) = 0$ for $u \in U$ and $\phi(v) = 1$ for $v \in V$. Clearly, ϕ is holomorphic on $U \cup V$. Note that $f(S)$, being the image of a compact set under a continuous map, is a compact subset of $U \cup V$. Then Runge's theorem implies that there is a sequence $\{p_n\}_{n=1}^{\infty}$ of polynomials which converges uniformly to ϕ on $f(S)$. Hence $\{p_n \circ f\}_{n=1}^{\infty}$ converges uniformly to $\phi \circ f$ on S. Note that $p_n \circ f \in A$ for all $n \in \mathbb{N}$ since $f \in A$. It then follows that $\phi \circ f|S \in \overline{A|S}$. Finally, note that $\phi \circ f|S$ is real-valued and $(\phi \circ f)(x) \neq (\phi \circ f)(y)$. ∎

Proof of Lemma 3.5. Suppose, on the contrary, that there is a point $x_0 \in L$ and a number $r_0 > 0$ such that $\mathcal{H}^2(L \cap \mathbb{B}(x_0, r_0)) = 0$. Note that $g\colon \mathfrak{M}_A \to \mathbb{R}$ defined by $g(x) = d(x, x_0)$ is a Lipschitz map with Lipschitz constant 1. We apply Eilenberg's inequality [16, Theorem 3.3.6] to obtain

$$\int^{*}_{(0,\infty)} \mathcal{H}^1(g^{-1}(t) \cap \mathbb{B}(x_0, r_0) \cap L)\, dt \leq \mathcal{H}^2(\mathbb{B}(x_0, r_0) \cap L) = 0,$$

where the integral is the upper integral on $(0, \infty)$. Consequently, $\mathcal{H}^1(g^{-1}(t) \cap \mathbb{B}(x_0, r_0) \cap L) = 0$ for almost every t in the interval $(0, r_0)$. Since $g^{-1}(t) \cap \mathbb{B}(x_0, r_0) = \mathbb{S}(x_0, t)$ for $0 < t < r_0$, we have $\mathcal{H}^1(\mathbb{S}(x_0, t) \cap L) = 0$ for almost every t in the interval $(0, r_0)$. Also, note that x_0 is not in K, a closed subset of $\mathfrak{M}_A$. Therefore, there exists s_0 in $(0, r_0)$ such that $K \cap \overline{\mathbb{B}(x_0, s_0)}$ is empty and $\mathcal{H}^1(\mathbb{S}(x_0, s_0) \cap L) = 0$. In particular, $K \cap \mathbb{S}(x_0, s_0)$ is empty, and hence $\mathbb{S}(x_0, s_0) \cap L = \mathbb{S}(x_0, s_0)$. Therefore, we get $\mathcal{H}^1(\mathbb{S}(x_0, s_0)) = 0$. Then Lemma 3.6 implies that the set $\mathbb{S}(x_0, s_0)$ is $\hat{A}$-convex. It now follows from Rossi's local maximum modulus principle [8, Ch. III, Theorem 8.2] that $\overline{\mathbb{B}(x_0, s_0)} \subseteq [(K \cap \mathbb{B}(x_0, s_0)) \cup \mathbb{S}(x_0, s_0)]^{\wedge}$. Consequently, $\overline{\mathbb{B}(x_0, s_0)} \subseteq \mathbb{S}(x_0, s_0)$

since $K\cap\overline{\mathbb{B}(x_0,s_0)}$ is empty and $\mathbb{S}(x_0,s_0)$ is $\hat{A}$-convex. This is a contradiction because x_0 is in $\overline{\mathbb{B}(x_0,s_0)}$, but not in $\mathbb{S}(x_0,s_0)$. ■

Let K be a subset of an m-dimensional manifold of class C^1 and $\mathcal{F}$ be a collection of C^1 functions on (varying neighborhoods of) K. The *exceptional set* $K_{\mathcal{F}}$ of K relative to $\mathcal{F}$ is defined by

$$K_{\mathcal{F}} = \{p \in K : df_1 \wedge \cdots \wedge df_m(p) = 0 \text{ for each } m\text{-tuple } f_1, \ldots, f_m \text{ in } \mathcal{F}\}.$$

LEMMA 3.7. *Let M be a real m-dimensional manifold of class C^2 and K be a compact subset of M. Suppose A is a uniform algebra on K generated by a collection of functions of class C^2 on (varying neighborhoods of) K in M. Also assume that*

(i) *the maximal ideal space of A is K, and*
(ii) *every point of K is isolated for A.*

Then the exceptional set $K_{\mathcal{F}}$ of K relative to $\mathcal{F} = A \cap C^2(M)$ has empty interior in M.

Let $\mathbb{D}$ denote the open unit disc in the complex plane. An *analytic disc* is a continuous injective map $\Phi\colon \overline{\mathbb{D}} \to \mathbb{C}^n$ which is holomorphic in $\mathbb{D}$. By the *boundary* of the analytic disc Φ, we will mean the map $\Phi|\partial\mathbb{D}$, the restriction of Φ to the unit circle $\partial\mathbb{D}$. Often in the literature, the analytic disc and its boundary are identified with their respective images in $\mathbb{C}^n$. In order to prove Lemma 3.7, we need the following result [13, Lemma 2.4] by Izzo concerning the existence of analytic discs.

LEMMA 3.8 ([13, Lemma 2.4]). *Let M be a real m-dimensional manifold of class C^2 and K be a compact subset of M. Suppose A is a uniform algebra on K generated by a collection $\mathcal{F}$ of functions of class C^2 on (varying neighborhoods of) K in M. If the exceptional set $K_{\mathcal{F}}$ has nonempty interior* $\operatorname{int}(K_{\mathcal{F}})$ *in M, then the maximal ideal space of A contains an analytic disc whose boundary is contained in* $\operatorname{int}(K_{\mathcal{F}})$.

Proof of Lemma 3.7. We claim that K contains no analytic disc. Suppose, on the contrary, that there is an analytic disc $\Phi\colon \overline{\mathbb{D}} \to K$. Then, for z in $\mathbb{D} \setminus \{0\}$, we obtain

$$\|z-0\|_{A(\overline{\mathbb{D}})} \geq \sup\{|f(\Phi(z)) - f(\Phi(0))| : f \in A,\ \|f\| \leq 1\} = \|\Phi(z) - \Phi(0)\|_A.$$

Since Φ is injective, by condition (ii), there is $\delta > 0$ with $\|\Phi(z)-\Phi(0)\|_A \geq \delta$ for all $z \in \mathbb{D} \setminus \{0\}$. Therefore, $\|z-0\|_{A(\overline{\mathbb{D}})} \geq \delta$ for all $z \in \mathbb{D} \setminus \{0\}$. In other words, $0 \in \overline{\mathbb{D}}$ is an isolated point (in the dual space norm) for $A(\overline{\mathbb{D}})$, a contradiction. Hence our claim is true, i.e., K does not contain any analytic disc.

Next suppose, on the contrary, that $K_{\mathcal{F}}$ has nonempty interior in M. Then by Lemma 3.8 and condition (i), K contains an analytic disc whose boundary is contained in $\mathrm{int}(K_{\mathcal{F}})$. However, this contradicts the fact that K does not contain any analytic disc. Consequently, $K_{\mathcal{F}}$ has empty interior in M. ∎

Let M be an abstract manifold and U be an open subset of M. A closed subset V of U is a *real-analytic subvariety* of U if for each point p in V there exists a neighborhood $W \subseteq U$ of p in M and a finite collection $\mathcal{F}$ of real-valued functions that are real-analytic in W such that

$$V \cap W = \{q \in W : f(q) = 0 \text{ for all } f \text{ in } \mathcal{F}\}.$$

A point p in V is a *regular point* (of *dimension* d) of V if there is a neighborhood O of p in M such that $V \cap O$ is a real-analytic submanifold (of dimension d) of O. A point of V that is not a regular point is a *singular point* of V. The set of all regular points of V is denoted by V_{reg}, whereas the set of all singular points of V is denoted by V_{sing}. The *dimension* of V is the largest integer d such that V has regular points of dimension d.

The following result concerns the Hausdorff measure of the singular set of a real-analytic subvariety of $\mathbb{C}^n$.

Lemma 3.9 ([7, Section 3.4.10]). *Let V be an m-dimensional real-analytic subvariety of an open subset U of $\mathbb{C}^n$. Then $\mathcal{H}^{m-1}(V_{\mathrm{sing}} \cap C)$ is finite for each compact subset C of U.*

4. Isolated point theorems. We first prove Theorem 1.2.

Proof of Theorem 1.2. First taking $K = M$ in Lemma 3.7, we deduce that the exceptional set $M_{\mathcal{F}}$ of M relative to $\mathcal{F} = A \cap C^2(M)$ has empty interior in M. Then taking $E = M_{\mathcal{F}}$ in Theorem 3.3 we see that the essential set for A is contained in $M_{\mathcal{F}}$. Therefore, the essential set for A has empty interior in M. ∎

Finally, we present a proof of our main result.

Proof of Theorem 1.1. Let

$$E = \{p \in X : df_1 \wedge df_2 \wedge df_3(p) = 0 \text{ for all } f_1, f_2, f_3 \in \mathcal{F}\}$$

be the exceptional set of X relative to $\mathcal{F}$. Also let X_0 be the interior of X relative to M. Define $\tilde{E} = E \cap X_0$ and $K_0 = \partial X \cup \tilde{E}$ $(= \partial X \cup E)$. We note that K_0 is compact because ∂X and E are both closed subsets of the compact set X. Then Theorem 3.3 implies that K_0 contains the essential set for A. Therefore, the maximal ideal space of $A|K_0$ is K_0 by condition (i) and Theorem 3.2. Thus to show $A = C(X)$, it suffices to prove $A|K_0 = C(K_0)$.

It easily follows that $\tilde{E}$ is a real-analytic subvariety of X_0. Let $\tilde{E}_{\mathrm{c}} = \{p \in \tilde{E}_{\mathrm{reg}} : df_1 \wedge df_2(p) = 0$ (as a form on $\tilde{E}_{\mathrm{reg}}$) for all $f_1, f_2 \in \mathcal{F}\}$ and set $Z = \partial X \cup \tilde{E}_{\mathrm{sing}} \cup \tilde{E}_{\mathrm{c}}$. Then Z is a compact subset of K_0, and $K_0 \setminus Z$ $(= \tilde{E}_{\mathrm{reg}} \setminus \tilde{E}_{\mathrm{c}})$ is a two-dimensional manifold. Applying Theorem 3.3, we see that Z contains the essential set for $A|K_0$. Therefore, the maximal ideal space of $A|Z$ is Z by Theorem 3.2. Thus to prove $A|K_0 = C(K_0)$, it suffices to show that $A|Z = C(Z)$.

In order to show $A|Z = C(Z)$, we will apply Lemma 3.4 by taking for X in the lemma the set ∂X and for Y the set $\tilde{E}_{\mathrm{sing}} \cup \tilde{E}_{\mathrm{c}}$. First we verify that $\mathcal{H}^2(\tilde{E}_{\mathrm{sing}} \cup \tilde{E}_{\mathrm{c}}) = 0$. To prove this, we introduce the following metric on M. Using the Whitney embedding theorem (see [12, Ch. 1, §8]) we embed M into $\mathbb{R}^n$ for some natural number n and then consider the metric on M inherited from $\mathbb{R}^n$ (with Euclidean metric). As a result, every coordinate system on M is a lipeomorphism (i.e., bi-Lipschitz) on compact subsets. Hence every real-analytic function on an open subset of M is Lipschitz on compact subsets. Moreover, $\mathcal{H}^n(W) = 0$ for a compact subset W of M if and only if $\mathcal{H}^n(\phi(W \cap U)) = 0$ for each coordinate chart (U, ϕ).

Note that $\tilde{E}$ is a real-analytic subvariety of X_0. Then by Lemma 3.7, E and hence $\tilde{E}$ has empty interior in M. Therefore, the dimension of $\tilde{E}$ is at most two. Now to show that $\mathcal{H}^2(\tilde{E}_{\mathrm{sing}}) = 0$, fix a coordinate chart (U, ϕ). Then the set $\phi(\tilde{E} \cap U)$ is a real-analytic subvariety of $\phi(U)$ with singular set $\phi(\tilde{E}_{\mathrm{sing}} \cap U)$. Therefore, by Lemma 3.9, $\mathcal{H}^1(\phi(\tilde{E}_{\mathrm{sing}} \cap U) \cap C)$ is finite for every compact subset C of $\phi(U)$. Now covering U by countably many compact sets, we obtain $\mathcal{H}^2(\phi(\tilde{E}_{\mathrm{sing}} \cap U)) = 0$. Consequently, $\mathcal{H}^2(\tilde{E}_{\mathrm{sing}}) = 0$. Next, note that $\tilde{E}_{\mathrm{c}}$ is a real-analytic subvariety of $\tilde{E}_{\mathrm{reg}}$. To show $\mathcal{H}^2(\tilde{E}_{\mathrm{c}}) = 0$, fix a point p in $\tilde{E}_{\mathrm{reg}}$. Since $\tilde{E}_{\mathrm{reg}}$ is open in $\tilde{E}$ and $\tilde{E}$ is open in K_0, clearly $\tilde{E}_{\mathrm{reg}}$ is open in K_0. Therefore, there is $r > 0$ such that $\overline{\mathbb{B}(p,r)} \cap K_0 \subseteq \tilde{E}_{\mathrm{reg}}$. Denote $\overline{\mathbb{B}(p,r)} \cap K_0$ by K_p. Then K_p is a compact subset of $\tilde{E}_{\mathrm{reg}}$. Note that the set K_p, being an intersection of two A-convex sets, is A-convex. Also, by Lemma 3.1(ii), every point of K_0 is an isolated point for $A|K_0$. Therefore, Lemma 3.7 implies that $\tilde{E}_{\mathrm{c}} \cap K_p$ has empty interior in $\tilde{E}_{\mathrm{reg}}$. Since the point p in $\tilde{E}_{\mathrm{reg}}$ is arbitrary, $\tilde{E}_{\mathrm{c}}$ is a real-analytic subvariety of $\tilde{E}_{\mathrm{reg}}$ of dimension at most one. Consequently, $\mathcal{H}^2(\tilde{E}_{\mathrm{c}}) = 0$. Thus, $\mathcal{H}^2(\tilde{E}_{\mathrm{sing}} \cup \tilde{E}_{\mathrm{c}}) = 0$.

Next, we verify that $\overline{A|\partial X} = C(\partial X)$. We will first show that the maximal ideal space of $\overline{A|\partial X}$ is ∂X. Note that the A-convex hull $\widehat{\partial X}$ of ∂X is contained in Z as ∂X is a subset of an A-convex set Z. Therefore, $\widehat{\partial X} \setminus \partial X$ is contained in $\tilde{E}_{\mathrm{sing}} \cup \tilde{E}_{\mathrm{c}}$, whose two-dimensional Hausdorff measure is zero. Then from Lemma 3.5 we see that $\widehat{\partial X} \setminus \partial X$ is empty, i.e., ∂X is A-convex. Hence the maximal ideal space of $\overline{A|\partial X}$ is ∂X. Next, by Lemma 3.1(ii), every point of ∂X is an isolated point for $\overline{A|\partial X}$. Then an application of the two-dimensional isolated point theorem [10, Theorem 2.1] yields $\overline{A|\partial X} = C(\partial X)$.

Finally, applying Lemma 3.4 we obtain $A|Z = C(Z)$ and hence the desired conclusion $A = C(X)$. ∎

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